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Letter to the Editor

On an entry of Ramanujan in his Notebooks: a nested roots expansion

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Abstract

In this letter, the elementary result of Ramanujan for nested roots, also called continued or infinite radicals, for a given integer N, expressed by him as a simple sum of three parts ($N = x + n + a$) is shown to give rise to two distinguishably different expansion formulas. One of these is due to Ramanujan and surprisingly, it is this other formula, not given by Ramanujan, which is more rapidly convergent!

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MSC: 01A32; 01A60; 17C17

Srinivasa Ramanujan (1887–1920), the 20th century Indian mathematical genius, left behind three Notebooks with more than 3000 entries, noted down between 1903 and 1912—when he was in search of a benefactor and recognition for his mathematical discoveries—and a ‘Lost’ Notebook, containing an additional 600 theorems, during the last year of his life (March 1919–April 1920), when he was fatally ill. His earliest mathematical contributions were in the form of Questions and/or Answers to Questions in the Journal of the Indian Mathematical Society (JIMS). He made 58 such contributions to JIMS, including the celebrated Rogers–Ramanujan identities.

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Ramanujan posed the following as Question 289, in the JIMS [10]:
289. (S. Ramanujan).—Find the value of:

(i) $1+\sqrt{(1+2\sqrt{(1+3\sqrt{(1+\&c\cdots\cdots)})})}$

(ii) $1+\sqrt{(6+2\sqrt{(7+3\sqrt{(8+\&c\cdots\cdots)})})}$.

Ramanujan also provided [11] the following solution to this Q. 289:

(i) Notice that $n(n+2)=n\sqrt{1+(n+1)(n+3)}$. Let $f(n)=n(n+2)$, then,

$$f(n)=n\sqrt{1+f(n+1)}=n\sqrt{1+(n+1)\sqrt{1+f(n+2)}}=\cdots,$$

that is

$$n(n+2)=n\sqrt{1+(n+1)\sqrt{1+(n+2)\sqrt{1+\cdots}}}.$$

Putting $n=1$, we have $\sqrt{1+2\sqrt{1+3\sqrt{1+\cdots}}}=3$.

(ii) In a similar manner: $n(n+3)=n\sqrt{(n+5)+(n+1)(n+4)}$. Supposing $f(n)=n(n+3)$, we have

$$f(n)=n\sqrt{(n+5)+f(n+1)}=n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+f(n+2)\cdots}}=\cdots$$

thus

$$n(n+3)=n\sqrt{(n+5)+(n+1)\sqrt{(n+6)+(n+2)\sqrt{(n+7)+\cdots}}}.$$

Putting $n=1$, we have $\sqrt{6+2\sqrt{7+3\sqrt{8+\cdots}}}=4$.

Ramanujan noted down the general result as Entry 4, in Chapter XIV of his first Notebook [14] and as Entry 4, in Chapter XII of his second Notebook [15], which reads

$$4.\quad x+n+a=\sqrt{ax+(n+a)^2+x\sqrt{a(x+n)+(n+a)^2+(x+n)\sqrt{\&c}}},\qquad (*)$$

$$\text{e.g. (i)}\quad 3=\sqrt{1+2\sqrt{1+3\sqrt{1+4\sqrt{1+\&c}}}},$$

$$\text{(ii)}\quad 4=\sqrt{6+2\sqrt{7+3\sqrt{8+4\sqrt{9+\&c}}}}.$$

The examples are obtained from (*) by putting $x=2$, $n=1$ and $a=0$ and 1, respectively.

Bruce C. Berndt [3], in his work on *Ramanujan's Notebooks*, writes down this Entry 4 as
Entry 4. *Let a, n and x denote arbitrary complex numbers. Then*

$$\begin{aligned} f(x) &:= x+n+a \\ &= (ax+(n+a)^2+x(a(x+n)+(n+a)^2+(x+n)(a(x+2n) \\ &\quad +(n+a)^2+(x+2n)(\cdots)^{1/2})^{1/2})^{1/2})^{1/2}. \end{aligned} \tag{1}$$

Proof. By successively substituting, we find that

$$\begin{aligned} f(x) &= (ax + (n+a)^2 + xf(x+n))^{1/2} \\ &= (ax + (n+a)^2 + x(a(x+n) + (n+a)^2 + (x+n)f(x+2n))^{1/2})^{1/2} \\ &= \cdots, \end{aligned}$$

and therefore we obtain the proposed formula. □

Examples. We have

(i) $3 = (1 + 2(1 + 3(1 + 4(1 + \cdots)^{1/2})^{1/2})^{1/2})^{1/2}$

and

(ii) $4 = (6 + 2(7 + 3(8 + 4(9 + \cdots)^{1/2})^{1/2})^{1/2})^{1/2}$.

In a recent article, Alexander Abian and Sergei Sverchkov [1], surprisingly without any reference to the work of Ramanujan, give an induction proof for a special case of Ramanujan's theorem, which is stated as

Theorem. *For every real number $x \geqslant 0$ it is the case that*

$$1 + x = \lim_{n\to\infty} \sqrt{1 + x\sqrt{1 + (1+x)\sqrt{1 + (2+x)\sqrt{\cdots\sqrt{1 + (n+x)\sqrt{1 + (n+1+x)}}}}}}. \tag{2}$$

For every non-negative real number, considered as made of up to three parts, Ramanujan gave a special representation for $N = x + n + a$, as the limit of an infinite iteration of square roots. This result gives several possible nested root representations for non-negative integers. Ramanujan's result can be stated in the form of the following theorem:

Theorem. *For every non-negative integer* $\geqslant 0$,

$$\begin{aligned} x_1 + x_2 + x_3 = \lim_{m\to\infty} & [x_2 x_1 + (x_2 + x_3)^2 + x_1[x_2(x_1 + x_3) + (x_2 + x_3)^2 \\ & + (x_1 + x_3)[x_2(x_1 + 2x_3) + (x_2 + x_3)^2 + (x_1 + 2x_3) \\ & \times [\cdots [x_2(x_1 + (m-1)x_3) + (x_2 + x_3)^2 \\ & + (x_1 + (m-1)x_3)(x_2 + x_3)]^{1/2}]^{1/2}]^{1/2}]^{1/2}]^{1/2}. \end{aligned} \tag{3}$$

Proof. Consider a sequence of functions $f_m(x_1, x_2 + x_3)$, for $(x_1, x_2, x_3 \geqslant 0)$, which satisfy

$$f_m(x_1, x_2 + x_3) = \sqrt{x_2 x_1 + (x_2 + x_3)^2 + x_1 f_{m-1}(x_1 + x_3, x_2 + x_3)}, \tag{4}$$

for all $(m = 1, 2, 3, \ldots)$ and

$$f_0(-, x_2 + x_3) = x_2 + x_3. \tag{5}$$

The recurrent use of (5) establishes (4). It suffices to prove that

$$x_1 + x_2 + x_3 = \lim_{m\to\infty} f_m(x_1, x_2 + x_3). \tag{6}$$

If we assume

$$f_{m-1}(x_1, x_2 + x_3) \leqslant x_1 + x_2 + x_3, \tag{7}$$

then from (5) it follows that

$$\begin{aligned} f_m(x_1, x_2 + x_3) &\leqslant \sqrt{x_2 x_1 + (x_2 + x_3)^2 + x_1(x_1 + x_2 + 2x_3)}, \\ &= x_1 + x_2 + x_3. \end{aligned} \tag{8}$$

From (9), it follows that

$$0 \leqslant x_1 + x_2 + x_3 - f_m(x_1, x_2 + x_3). \tag{9}$$

Multiply and divide by $x_1 + x_2 + x_3 + f_m(x_1, x_2 + x_3)$ to get for (10)

$$0 \leqslant \frac{(x_1 + x_2 + x_3)^2 - f_m^2(x_1, x_2 + x_3)}{x_1 + x_2 + x_3 + f_m(x_1, x_2 + x_3)}. \tag{10}$$

Since, $(x_1, x_2, x_3 \geqslant 0)$, consistent with (8) and the result (7) to be proved, we assume

$$f_m(x, x_2 + x_3) \geqslant x_3 - x_2, \quad m > 0, \text{ for all } x, \tag{11}$$

so that, (11) becomes, using (12) for the denominator and (9) for the numerator,

$$\begin{aligned} 0 &\leqslant \frac{(x_1 + x_2 + x_3)^2 - [x_2 x_1 + (x_2 + x_3)^2 + x_1 f_{m-1}(x_1 + x_3, x_2 + x_3)]}{x_1 + 2x_3} \\ &= \frac{x_1}{x_1 + 2x_3}[x_1 + x_2 + 2x_3 - f_{m-1}(x_1 + x_3, x_2 + x_3)]. \end{aligned} \tag{12}$$

In successive steps, continuing to multiply and divide by the factors $x_1 + x_2 + 2x_3 + f_{m-1}(x_1 + x_3, x_2 + x_3), x_1 + x_2 + 3x_3 + f_{m-2}(x_1 + 2x_3, x_2 + x_3), \ldots, x_1 + x_2 + nx_3 + f_1(x_1 + (m-1)x_3, x_2 + x_3)$, after simplifications, using (6) in the last step, we get

$$\begin{aligned} 0 &\leqslant \frac{x_1}{x_1 + 2x_3} \frac{x_1 + x_3}{x_1 + 3x_3} \frac{x_1 + 2x_3}{x_1 + 4x_3} \cdots \frac{x_1 + (m-1)x_3}{x_1 + (m+1)x_3} \cdot (x_1 + mx_3) \\ &= \frac{x_1(x_1 + x_3)}{x_1 + (m+1)x_3}, \end{aligned} \tag{13}$$

for every $(x_1, x_2, x_3 \geqslant 0)$ and $m \geqslant 0$. Clearly, in the limit $m \to \infty$,

$$\lim_{m\to\infty} \frac{x_1(x_1 + x_3)}{x_1 + (m+1)x_3} = 0. \tag{14}$$

Eq. (14) implies (7), and hence (4), which completes the proof. Here, we adapted the proof of Abian and Sverchkov to prove Ramanujan's Entry (*), for an integer expressed as $x_1 + x_2 + x_3$.

Setting, in (4), $x_1 = x$, $x_2 = a$, $x_3 = n$, we get the Entry (*) of Ramanujan. The result (3) of Abian and Sverchkov is obtained as a special case of the Entry (*) of Ramanujan, for $x_1 = x$, $x_2 = a = 0$, $x_3 = n = 1$, in (4). From Ramanujan's Entry, for $x_1 = x,\ x_2 = a = 1,\ x_3 = n = 0$, in (4), we obtain, instead of (3) the expansion

$$1 + x = \sqrt{1 + x + x\sqrt{1 + x + x\sqrt{1 + x + x\sqrt{1 + x + \cdots}}}}. \tag{15}$$

Below, we list the different nested root representations for 2 and 3 given by Ramanujan's Entry (*). We have also computed numerically, using Maple and Mathematica, the values of the nested roots, if the nesting is up to 10 roots only, to give an idea of the nature of convergence of the different nested root representations.

$$\begin{aligned}
2 \quad &= 2 + 0 + 0 = (2(2(2(\cdots)^{1/2})^{1/2})^{1/2} && = 1.99864665500530\cdots\\
&= 1 + 0 + 1 = (1 + 1(1 + 2(1 + 3\cdots(1 + 10)^{1/2})^{1/2})^{1/2} && = 1.99747850066804\cdots\\
&= 1 + 1 + 0 = (2 + (2 + (2 + \cdots)^{1/2})^{1/2})^{1/2} && = 1.99999895417917\cdots,\\
3 \quad &= 3 + 0 + 0 = (3(3(3(\cdots)^{1/2})^{1/2})^{1/2} && = 2.99678313524759\cdots\\
&= 2 + 0 + 1 = (1 + 2(1 + 3(1 + 4\cdots(1 + 11)^{1/2})^{1/2})^{1/2} && = 2.99480026926620\cdots\\
&= 2 + 1 + 0 = (3 + 2(3 + 2(3 + 2\cdots)^{1/2})^{1/2})^{1/2} && = 2.99995845129357\cdots\\
&= 1 + 0 + 2 = (4 + 1(4 + 3(4 + 5\cdots(4 + 19)^{1/2})^{1/2})^{1/2} && = 2.99691068883683\cdots\\
&= 1 + 2 + 0 = (6 + (6 + (6 + \cdots)^{1/2})^{1/2})^{1/2} && = 2.99999996442454\cdots\\
&= 1 + 1 + 1 = (5 + 1(6 + 2(7 + 3\cdots(14 + 10)^{1/2})^{1/2})^{1/2} && = 2.99965789054637\cdots.
\end{aligned}$$

Note that when $x_1 = 0$, the nested roots expression of Ramanujan gives the trivial result that $n + a = \sqrt{(n + a)^2}$. The nested root formula of Ramanujan for 3 will not produce the following representations:

$$\begin{aligned}
3 \quad &= \sqrt{7 + 2}\\
&= (7 + (2(2(2(\cdots)^{1/2})^{1/2})^{1/2} && = 2.99954900373071\cdots\\
&= (7 + (1 + 1(1 + 2(1 + 3\cdots 7(1 + 9)^{1/2})^{1/2})^{1/2} && = 2.99918864269139\cdots\\
&= (7 + (2 + (2 + (2 + \cdots(3)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} && = 2.99999930278621\cdots,
\end{aligned}$$

where we have used the three nested root representations for 2 given above as 2+0+0, 1+0+1 and 1+1+0, respectively, so that 3 is considered as the number being made up of four (instead of three) parts.

The question therefore arises why Ramanujan considered partitions of an integer into only three parts and that too, assuming the functional form as $f(x_1, x_2 + x_3)$, rather than the more general $f(x_1, x_2, x_3)$. For, in this general case, the recurrence relation would be

$$\lim_{n\to\infty} f_n(x_1, x_2, x_3) = x_1 + x_2 + x_3 \tag{16}$$

$$= \sqrt{x_1 x_2 + (x_2 + x_3)^2 + x_1 f_{n-1}(x_1, x_2, 2x_3)}. \tag{17}$$

Repeated use of this recurrence relation gives the nested root representation

$$x_1 + x_2 + x_3 = [x_1x_2 + (x_2 + x_3)^2 + x_1[x_1x_2 + (x_2 + 2x_3)^2 + x_1[x_1x_2 + (x_2 + 4x_3)^2 + x_1[x_1x_2 + (x_2 + 8x_3)^2 + x_1[\cdots]^{1/2}]^{1/2}]^{1/2}]^{1/2}]^{1/2} \tag{18}$$

which is distinctly different from the Entry (*) of Ramanujan. Setting $x_1 = x,\ x_2 = 1,\ x_3 = 0$, we get (16) and for $x_1 = x,\ x_2 = 0,\ x_3 = 1$, we get yet another different expansion

$$1 + x = \sqrt{1 + x\sqrt{4 + x\sqrt{16 + x\sqrt{64 + \cdots}}}}. \tag{19}$$

This nested roots formula (19), gives rise to newer expansions for the examples considered above, viz. 2 and 3. For 2, the cases 2+0+0 and 1+1+0 are the same as before and for 3, the cases 3+0+0, 2+1+0 and 1+2+0 are the same as before. The expansions that are different in these examples and more rapidly convergent than the ones given above are

$$\begin{aligned}
2 &= 1+0+1 = (1 + (4 + (16 + (64 + (256 + \cdots (65536 + (262145)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} && = 1.99999999999999402\cdots,\\
3 &= 2+0+1 = (1 + 2(4 + 2(16 + 2(64 + \cdots 2(65536 + 2(262146)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} && = 2.99999999999795156\cdots\\
&= 1+0+2 = (4 + (16 + (64 + \cdots (262144 + (1048577)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} && = 2.99999999999999997\cdots\\
&= 1+1+1 = (5 + (10 + (26 + (82 + \cdots (66050 + (263171)^{1/2})^{1/2})^{1/2})^{1/2})^{1/2})^{1/2} && = 2.99999999999999800\cdots.
\end{aligned}$$

An interesting discussion about the convergence of infinite nested roots, can be found in the article of Herschfeld [8], who proved that Ramanujan's solutions (examples (i) and (ii) of Q.289) are monotonic and converge to the limits 3 and 4, respectively. He derived the necessary and sufficient conditions for the convergence of the sequence $\{u_n\}$:

$$u_n \equiv \sqrt{a_1 + \sqrt{a_2 + \cdots + \sqrt{a_n}}} \tag{20}$$

and showed that the sequence $\{u_n\}$ converges iff there exists a finite upper limit:

$$\overline{\lim_{n\to\infty}}\, a_n^{2^{-n}} < +\infty. \tag{21}$$

Berndt points out [3] that Vijayaraghavan has indicated to Hardy [7] that

$$(a_1 + (a_2 + (a_3 + \cdots + (a_n)^{1/2})^{1/2})^{1/2})^{1/2}, \quad a_n \geqslant 0, \tag{22}$$

tends to a limit as $n \to \infty$, iff

$$\overline{\lim_{n\to\infty}}\, \frac{\operatorname{Log} a_n}{2^n} < \infty. \tag{23}$$

Allen [2] considered continued radicals of the form

$$u_n \equiv \sqrt[r]{a + \sqrt[r]{a + \sqrt[r]{a + \cdots}}}, \tag{24}$$

where r is an integer greater than 1 and a is any positive number. ($r = 2$ in (24) corresponds to (20)). Allen showed that u_n is less than the unique positive root of: $x^r - x - a = 0$ and the sequence $\{u_n\}$ converges to a limit. The interested reader may look into the Entry 5 of Ramanujan on general nested roots with $r > 2$, contained in his second notebook ([15, Chapter XII]. Also, cf. Berndt [3, p. 109]).

In fine, this is an example of how even in elementary mathematics, (in the 58 problems posed by Ramanujan), such as:

- the nested roots problem of Ramanujan [10,11];
- the Brocard [5]–Ramanujan [12] Diophantine equation: $n! + 1 = m^2$ which has till date only three known integer solutions corresponding to $n = 4, 5, 7$ and $m = 5, 11, 71$—a recent computer search by Berndt and Galway [4] has shown that there are no more solutions up to $n = 10^9$;
- the Nagell [9]–Ramanujan [13] equation: $x^2 + 7 = 2^n$ which has only five known integer solutions corresponding to $x = 1, 3, 5, 11, 181$ and $n = 3, 4, 5, 7, 15$—which led Yann Bugeaud and Shorey [6] to prove that there are no integer solutions for the Diophantine equation: $x^2 + 7 = 4y^n$, for $x \geqslant 1, y > 2, n > 1$;

and more than 3250 entries made by Ramanujan in his celebrated Notebooks, extensively studied by Berndt [3], "*Much work still needs to be done*" (Berndt, Preface in Ref. [3]).

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